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On the uniqueness of the Segal quantisation of linear Bose systems

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Abstract. The kinematical description of a classical system having a linear phase space can be quantised if a complexification is given over the symplectic space that supports the description; indeed, a complexification determines a Segal quantisation. The question of the uniqueness of the Segal quantisation arises since the quantum descriptions constructed through different Segal quantisations need not be equivalent. If the classical description contains also a symmetry group (represented by linear symplectic transformations; this means that we deal with systems having linear dynamics only), we can require the complexification to determine a Segal quantisation that also quantises the symmetry group. We show that this requirement determines uniquely the complexification, and therefore the Segal quantisation, provided the symmetry group fulfils a (real) irreducibility condition. We apply this uniqueness criterion to the free neutral scalar field.

1. Introduction

The connection between the classical description of a physical system and its quantum theoretical one has been investigated from many points of view.

A formally well defined approach to this problem, at least for the classical systems with a linear phase space, consists in using a linear symplectic space as the basic framework of the classical description and in performing the quantisation by means of a so-called Weyl system; this associates (for precise definitions see below) with each point of the symplectic space of the classical description a unitary operator on the Hilbert space of the quantum one. In this way an algebra can be generated, called the Weyl algebra, which, at least if suitable localisability requirements are introduced, is usually interpreted as the algebra of observables of the quantum system.

It is well known that:

(i) when the symplectic space is finite dimensional the existence problem for a Weyl system gets completely settled by the 'Schrödinger representation of the canonical commutation relations (CCR)'; the uniqueness problem is also settled by the von Neumann result, at least from a kinematical point of view; for what might be called a dynamical non-uniqueness (or a non-equivalence of different representations) see Gallone and Sparzani (1979).

(ii) For infinite dimensional systems the existence problem is not completely solved in the general setting given above. It is well known, though, that if one starts with a complex Hilbert space and looks at its underlying symplectic structure as the fundamental structure, then there are standard ways of building up Weyl systems, and hence representations of the CCR; e.g. the Fock–Cook space is constructed and second quantisation—also called Segal quantisation (Reed and Simon 1975)—provides an explicit representation; it is also well known that there is a very heavy non-uniqueness here unless strong conditions are imposed.

A crucial point where non-uniqueness may, and does, enter into the picture comes when one, in order to be able to follow a pattern such as the one mentioned in (ii), tries to look at the symplectic space as a complex Hilbert space. It is clear that the only possible meaning of the last sentence is that one has to graft into the real symplectic structure a complex Hilbert one, and this is accomplished essentially by defining what is to be the scalar multiplication by the imaginary unit (complexification).

In this paper we consider a possible situation in which the choice of the complexification is unique. Indeed, we consider the special class of the physical systems whose symmetry groups are classically represented by linear symplectic transformations (in particular their dynamic evolutions are linear). For such systems, conditions for the symmetry groups can be sought that lead to a unique complexification. A neat result has been obtained by Kay (1979), who proved that the requirement that the 'first quantised' energy be strictly positive is sufficient to ensure the uniqueness of the complexification.

In this paper we look for a condition that, being concerned with a classical description of the system, naturally enough bears upon the purely classical side of the symmetry group, i.e. its being represented by linear symplectic transformations, without any quantum-like request (as for instance the positivity of the 'first quantised' energy)†. What we find is that the (real) irreducibility of the mentioned representation implies the uniqueness of the complexification. Note that the complex irreducibility (which anyway could be defined only after the complexification has been set in) would not be sufficient, as trivial examples show.

It is worth pointing out explicitly that the uniqueness problem we are considering here concerns the Segal quantisation, i.e. the uniqueness of the complexification, and not the general problem of the quantisation of the physical systems with a linear dynamics. For results concerning the uniqueness (and the lack of uniqueness) of the general (algebraic) quantisation of these systems, see e.g. Segal (1962), Weinless (1969).

2. Weyl systems

A linear (to be understood in what follows) symplectic space is a pair $(\mathcal{M}, B)$, where $\mathcal{M}$ is a real vector space and B is a symplectic form on $\mathcal{M}$, i.e. a non-degenerate antisymmetric bilinear real form on $\mathcal{M}$. An $\mathcal{H}$-valued Weyl system over a symplectic space $(\mathcal{M}, B)$ is a map W from $\mathcal{M}$ into the group $\mathrm{U}(\mathcal{H})$ of the unitary automorphisms of a complex separable Hilbert space $\mathcal{H}$, satisfying:

(a) $\forall m, m' \in \mathcal{M}$, $W(m)W(m') = \exp(\frac{1}{2}\mathrm{i}B(m, m'))W(m+m')$,

(b) the map $\mathbb{R} \ni t \mapsto W(tm) \in \mathrm{U}(\mathcal{H})$ is weakly continuous, $\forall m \in \mathcal{M}$.

† It must be understood, however, that not every symplectic space is suited for Segal quantisation: as very little is known on the problem of the existence of complexification operators (see below) for a general symplectic space, we must at least demand that the symplectic space we use admits a complexification operator. Further, the number of the *a priori* classically admissible symplectic spaces with complexification operators is reduced by the choice of what the complexification operator should look like. And this is a 'request' that may stem from a sort of fore-knowledge of what we want as a quantised system. See also the footnote in § 4.

For Weyl systems, unitary equivalence and irreducibility are defined in the natural way. Indeed, two Weyl systems W and W' over the same symplectic space $(\mathscr{M}, B)$ are said to be unitarily equivalent if a unitary map V exists from the Hilbert space of W onto the Hilbert space of W' such that $\forall m \in \mathscr{M}$, $VW(m)V^{-1} = W'(m)$. Also, a Weyl system W is said to be irreducible if in its Hilbert space no non-trivial closed linear subspaces exist that are left invariant by the range of W.

If a Weyl system W is given over a symplectic space $(\mathscr{M}, B)$, a C^* algebra of operators along with a mapping from $\mathscr{M}$ into it can be constructed in an essentially unique way as follows. Let V be a finite dimensional subspace of $\mathscr{M}$ and let $\mathscr{A}_V$ be the W^* algebra generated by the image with respect to W of V. The norm-closure $\mathscr{A}(W)$ of $\cup\{\mathscr{A}_V;\ V$ finite dimensional subspace of $\mathscr{M}\}$ is a C^* algebra of operators and W is a mapping from $\mathscr{M}$ into $\mathscr{A}(W)$. Also the following Segal's uniqueness theorem holds (Segal 1959, see also Slawny 1972): if W' is another Weyl system over $(\mathscr{M}, B)$, there exists a unique isomorphism τ between $\mathscr{A}(W)$ and $\mathscr{A}(W')$ that carries W into W', i.e. such that $\forall m \in \mathscr{M}$, $\tau(W(m)) = W'(m)$. This amounts to uniqueness up to isomorphisms of the pair $(\mathscr{A}(W), W)$, which is called the Weyl algebra over $(\mathscr{M}, B)$. It is worth pointing out that by no means does τ need to be unitarily implementable, i.e. it may happen that the Weyl systems W and W' are not unitarily equivalent (also if they are related by τ in the way seen before). Only if $\mathscr{M}$ is finite dimensional and W and W' are both irreducible (the statement can be modified so as to dispose of this condition, though) is it always true that τ is unitarily implementable; in this case, Segal's uniqueness result is simply a rephrasing of the von Neumann uniqueness theorem.

We shall assume that a symplectic space $(\mathscr{M}, B)$ is the proper framework for the kinematical description of a classical system. It is then natural to assume that a linear (to be understood in what follows) symmetry of the classical system is represented by a symplectic transformation of $(\mathscr{M}, B)$, i.e. a linear automorphism of $\mathscr{M}$ that preserves the form B. The group of such transformations will be denoted by $\mathrm{Aut}(\mathscr{M}, B)$.

We shall also assume that a Weyl system over $(\mathscr{M}, B)$ provides a quantisation of the classical kinematical description $(\mathscr{M}, B)$. Then a classical symmetry S also becomes quantised. Indeed, $W \circ S$ is still a Weyl system over $(\mathscr{M}, B)$ because $S \in \mathrm{Aut}(\mathscr{M}, B)$. Thus, owing to Segal's uniqueness theorem, there is a unique automorphism τ_S of $\mathscr{A}(W)$ that carries $W(m)$ into $W(Sm)$ and the automorphism τ_S (which does not need to be unitarily implementable) is considered as the quantisation of the classical symmetry S.

The analysis of symmetry groups and their quantisations can now be made in a similar way. A classical symmetry group is a continuous symplectic representation in $(\mathscr{M}, B)$ of a Lie group G, i.e. a group homomorphism

$$\mathrm{G} \ni g \mapsto S_g \in \mathrm{Aut}(\mathscr{M}, B),$$

such that $\mathrm{G} \ni g \mapsto B(m, S_g m') \in \mathbb{R}$ is a continuous function, $\forall m, m' \in \mathscr{M}$. From the results obtained above for a symmetry, it follows that a group homomorphism $g \mapsto \tau_g$ of G into the group of the automorphisms of the C^* algebra $\mathscr{A}(W)$ exists such that $W(S_g m) = \tau_g(W(m))$. The classical symmetry group $g \mapsto S_g$ is naturally considered to be quantised by the homomorphism $g \mapsto \tau_g$. Indeed, this homomorphism is *the* quantisation of the classical symmetry group considered, since its algebraic uniqueness is an easy consequence of Segal's uniqueness theorem: if W' is another Weyl system over $(\mathscr{M}, B)$ and τ'_g the automorphism of $\mathscr{A}(W')$ such that $W'(S_g m) = \tau'_g(W'(m))$, then the relation $\tau_g = \tau^{-1}\tau'_g\tau$ holds if τ is the isomorphism of $\mathscr{A}(W)$ onto $\mathscr{A}(W')$ that carries W

into W'. Therefore, through Weyl systems we can quantise a classical symmetry group as a group of C^* algebra automorphisms in a way which is essentially unique.

The ultimate meaning of what precedes is as follows. Once a Weyl system W over $(\mathcal{M}, B)$ is given, it is possible to construct the Weyl algebra over $(\mathcal{M}, B)$, i.e. a C^* algebra $\mathcal{A}_{(\mathcal{M},B)}$ (any C^* algebra isomorphic with $\mathcal{A}(W)$) along with a labelling w of its elements by the elements of $\mathcal{M}$ (w is what W becomes through the isomorphism of $\mathcal{A}(W)$ with $\mathcal{A}_{(\mathcal{M},B)}$). Moreover, a classical symmetry group gets quantised through automorphisms of $\mathcal{A}_{(\mathcal{M},B)}$. Remarkably enough, all this can be done also if no Weyl system is given over $(\mathcal{M}, B)$, i.e. the Weyl algebra $(\mathcal{A}_{(\mathcal{M},B)}, w)$ can still be defined (Segal 1956) and a classical symmetry group can be quantised by automorphisms of $\mathcal{A}_{(\mathcal{M},B)}$. Indeed, a Hilbert space for the quantised system is derived from $\mathcal{A}_{(\mathcal{M},B)}$ using the GNS theorem, and the existence of a Weyl system over $(\mathcal{M}, B)$ is equivalent to the existence of a 'regular' state of $\mathcal{A}_{(\mathcal{M},B)}$ (Segal 1959, 1961, 1963). This shows how the quantisation of a classical description can be performed in purely algebraic terms, with no Hilbert space playing a fundamental role (the occurrence of Weyl systems is reduced to the existence of particular states of the Weyl algebra). Besides, also if a Weyl system W is being used and a classical symmetry group is being quantised in its framework through automorphisms τ_g of $\mathcal{A}(W)$, the automorphisms τ_g need not be unitarily implementable. Nevertheless, if a classical symmetry group is given over $(\mathcal{M}, B)$, it may be important to examine when it is possible to find a Weyl system W such that τ_g is unitarily implementable by means of a continuous unitary representation of G in the Hilbert space in which W is defined. Indeed, if this is the case, we are able to represent the Lie algebra of G by means of self-adjoint operators acting in the Hilbert space where the quantum kinematical description is given (Nelson 1959, Gårding 1960).

In other words, it appears that—also if the foundations of the theory of quantisation can be expressed in a purely algebraic framework—it can be desirable to be able to construct, over a symplectic space $(\mathcal{M}, B)$, a Weyl system and, for a classical symmetry group $\mathrm{G} \ni g \mapsto S_g \in \mathrm{Aut}(\mathcal{M}, B)$, a continuous unitary representation that quantises it. Namely, we are led to look for a Weyl system W over $(\mathcal{M}, B)$ and a continuous unitary representation U of G in the Hilbert space where W is defined such that $\forall m \in \mathcal{M}$, $\forall g \in \mathrm{G}$, $W(S_g m) = U(g)W(m)U(g)^{-1}$.

3. Segal quantisation

A method to construct Weyl systems and at the same time to quantise classical symmetry groups by continuous unitary representations will be explained in this section. First, we recall a few basic facts about second quantisation; for more details, see e.g. Reed and Simon (1975).

Let $\mathcal{K}$ be a complex separable Hilbert space and let $\mathcal{F}_s(\mathcal{K})$ be the symmetric Fock space over $\mathcal{K}$, i.e. $\mathcal{F}_s(\mathcal{K}) := \oplus_{n=0}^{\infty} \mathcal{K}_s^{(n)}$, where $\mathcal{K}_s^{(n)}$ is the n-fold symmetric tensor product of $\mathcal{K}$. Denoting by $a(k)$ and $a^+(k)$, with $k \in \mathcal{K}$, the annihilation and creation operators defined in the usual way in $\mathcal{F}_s(\mathcal{K})$, the operator $(1/\sqrt{2})(a(k)+a^+(k))$ is essentially self-adjoint; setting, for $k \in \mathcal{K}$,

$$W(k) := \exp[(\mathrm{i}/\sqrt{2})\overline{(a(k)+a^+(k))}]$$

(where the bar means closure), the mapping from $\mathcal{K}$ to the unitary automorphisms of $\mathcal{F}_s(\mathcal{K})$ given by $k \mapsto W(k)$ is called the Segal quantisation over $\mathcal{K}$ and satisfies the following conditions:

(a) $W(k)W(k') = \exp[-(i/2)\,\mathrm{Im}(k|k')]W(k+k')$, $\forall k, k' \in \mathscr{K}$,

(b) the map $\mathbb{R} \ni t \mapsto W(tk) \in \mathrm{U}(\mathscr{F}_s(\mathscr{K}))$ is weakly continuous, $\forall k \in \mathscr{K}$.

Also, if S is a unitary operator in $\mathscr{K}$ and $\Gamma(S)$ its second quantisation, the relation $\Gamma(S)W(k)\Gamma(S)^{-1} = W(Sk)$ holds, $\forall k \in \mathscr{K}$. Finally, if $g \mapsto S_g$ is a continuous unitary representation of a topological group G in $\mathscr{K}$, the map $\mathrm{G} \ni g \mapsto \Gamma(S_g) \in \mathrm{U}(\mathscr{F}_s(\mathscr{K}))$ is a continuous unitary representation of G in $\mathscr{F}_s(\mathscr{K})$.

Now, notice that $-\mathrm{Im}(k|k')$ is a symplectic form on the real vector space $\mathscr{K}$ (when we say so, we mean to consider in $\mathscr{K}$ the real vector space structure underlying the complex Hilbert space structure of $\mathscr{K}$); the pair $(\mathscr{K}, -\mathrm{Im}(\cdot|\cdot))$ will be called the symplectic space generated by the complex Hilbert space $\mathscr{K}$. Also, the Segal quantisation over $\mathscr{K}$ defined above is an $\mathscr{F}_s(\mathscr{K})$-valued Weyl system over $(\mathscr{K}, -\mathrm{Im}(\cdot|\cdot))$, as can be easily seen.

So, through Segal quantisation, a Weyl system can actually be constructed over a symplectic space $(\mathcal{M}, B)$ when a complex Hilbert space $\mathscr{K}$ can be found such that $(\mathcal{M}, B)$ is the symplectic space generated by $\mathscr{K}$. Moreover, since the restriction of a Weyl system to a real vector subspace is still a Weyl system, this procedure covers also the case when for the symplectic space $(\mathcal{M}, B)$ the following conditions hold:

(i) $\mathcal{M}$ is a real vector subspace of $\mathscr{K}$, which means that $\mathcal{M}$ can be identified with a subset of $\mathscr{K}$ and the real vector operations induced on $\mathcal{M}$ by $\mathscr{K}$ coincide with those already existing on $\mathcal{M}$,

(ii) $\forall m, m' \in \mathcal{M} \subset \mathscr{K}$, $B(m, m') = -\mathrm{Im}(m|m')$.

Let us now suppose, as it is in our opinion the most basic situation, we start with a real symplectic space $(\mathcal{M}, B)$ and also suppose that a (real) linear operator J is given on $\mathcal{M}$ which satisfies the following conditions: $J^2 = -\mathbb{1}$; $\forall m, m' \in \mathcal{M}$, $B(Jm, Jm') = B(m, m')$; $B(Jm, m) > 0$ iff $m \neq 0$. Such an operator J is called a complexification operator over $(\mathcal{M}, B)$ since by setting: $\alpha m := ((\mathrm{Re}\,\alpha)\mathbb{1} + (\mathrm{Im}\,\alpha)J)m$, $\forall \alpha \in \mathbb{C}$ (the complex plane), $\forall m \in \mathcal{M}$, and $(m|m') := B(Jm, m') - iB(m, m')$, $\forall m, m' \in \mathcal{M}$, a complex inner product space structure is defined over the set $\mathcal{M}$. Moreover, its completion $\mathcal{M}_J$ is a complex Hilbert space for which the conditions (i) and (ii) above hold with respect to $\mathcal{M}$, that is the symplectic space generated by $\mathcal{M}_J$ contains $(\mathcal{M}, B)$.

In conclusion, a way of constructing a Weyl system over a symplectic space $(\mathcal{M}, B)$ consists in finding a complexification operator J over $(\mathcal{M}, B)$, constructing the Segal quantisation over $\mathcal{M}_J$ and restricting it to $\mathcal{M}$. Denoting by $\mathscr{J}_{(\mathcal{M},B)}$ the family of the complexification operators over $(\mathcal{M}, B)$, nothing can in general be said about what $\mathscr{J}_{(\mathcal{M},B)}$ contains. Indeed, there is a bijection between $\mathscr{J}_{(\mathcal{M},B)}$ and the class of the complex Hilbert spaces such that the condition (ii) above holds along with

(i′) $\mathcal{M}$ is a dense complex vector subspace of the complex Hilbert space,

as is very easy to see. If, however, $\mathscr{J}_{(\mathcal{M},B)}$ is not empty, it usually contains plenty of complexification operators, since if $J \in \mathscr{J}_{(\mathcal{M},B)}$ and $S \in \mathrm{Aut}(\mathcal{M}, B)$, then $S^{-1}JS \in \mathscr{J}_{(\mathcal{M},B)}$. Of course, there is no need for two Weyl systems constructed through Segal quantisation from two different elements of $\mathscr{J}_{(\mathcal{M},B)}$ to be unitarily equivalent. Indeed there are criteria for this to happen (Shale 1962, Van Daele and Verbeure 1971).

From what was seen before, it is clear that in order for the quantisation of a classical symmetry, i.e an element S of $\mathrm{Aut}(\mathcal{M}, B)$, to be unitarily implementable within the quantum kinematical description set up by a Weyl system W defined through the Segal quantisation by a complexification operator J, it is enough that an extension of S exist to a unitary operator in the complex Hilbert space $\mathcal{M}_J$. For this to happen, it is necessary and sufficient that S commute with J. If this is the case, indeed, $\Gamma(\bar{S})$ (where $\bar{S}$ is the closure of S in the complex Hilbert space $\mathcal{M}_J$) is a unitary operator in $\mathscr{F}_s(\mathcal{M}_J)$ such that

$\forall m \in \mathcal{M}$, $\Gamma(\bar{S})W(m)\Gamma(\bar{S})^{-1} = W(Sm)$. Also, if $G \ni g \mapsto S_g \in \mathrm{Aut}(\mathcal{M}, B)$ is a classical symmetry group such that S_g commutes with J for each $g \in G$, the mapping $G \ni g \mapsto \bar{S}_g \in U(\mathcal{M}_J)$ is a continuous unitary representation, whence $G \ni g \mapsto \Gamma(\bar{S}_g) \in U(\mathcal{F}_s(\mathcal{M}_J))$ is a continuous unitary representation of G in $\mathcal{F}_s(\mathcal{M}_J)$ which quantises the classical symmetry group, i.e. such that $\forall m \in \mathcal{M}, \forall g \in G$, $W(S_g m) = \Gamma(\bar{S}_g)W(m)\Gamma(\bar{S}_g)^{-1}$.

If for a symplectic space $(\mathcal{M}, B)$ we are given the set $\mathcal{J}_{(\mathcal{M},B)}$ of the complexification operators and a classical symmetry group $g \mapsto S_g$, it may happen that for no $J \in \mathcal{J}_{(\mathcal{M},B)}$ do we have $[J, S_g] \equiv JS_g - S_g J = 0$, or that this happens for some elements of $\mathcal{J}_{(\mathcal{M},B)}$ or for just one element of $\mathcal{J}_{(\mathcal{M},B)}$. In a previous paper (Gallone and Sparzani 1979) a symplectic space was examined for which the classical symmetry groups divided into two classes, according to the existence of no or just one complexification operator that commuted.

In the following section we shall give a criterion for the uniqueness of a complexification operator that commutes with a classical symmetry group, granted its existence.

4. A theorem of uniqueness

As was explained in the previous sections, the kinematical description of a classical system given through a symplectic space $(\mathcal{M}, B)$ is quantised by the second (i.e. Segal) quantisation over $\mathcal{M}_J$ once a complexification operator over $(\mathcal{M}, B)$, i.e. an element J of $\mathcal{J}_{(\mathcal{M},B)}$, is given. Also, the Segal quantisations that can be constructed starting from two elements of $\mathcal{J}_{(\mathcal{M},B)}$ need not be unitarily equivalent. If, however, the classical description we start from contains also a symmetry group, i.e. a homomorphism of a group into the group $\mathrm{Aut}(\mathcal{M}, B)$ of the symplectic transformations of $(\mathcal{M}, B)$, we can ask to what extent the requirement that the symmetry group be quantised by the second quantisation over $\mathcal{M}_J$ determines J within $\mathcal{J}_{(\mathcal{M},B)}$. We shall presently show that this requirement determines J uniquely, provided the symmetry group fulfils a (real Hilbert space) irreducibility condition.

Using this criterion, we shall see that the symplectic space of the real solutions of the positive mass μ and spin zero Klein–Gordon equation, where the symplectic form is the Wronskian of two solutions and the symmetry group is the Poincaré group, admits of a unique complexification operator. The action of the Poincaré group we assume is the obvious one $\phi(x) \mapsto \phi(\Lambda^{-1}(x-a))$, where ϕ is a solution and (a, Λ) an element of the group. The quantum system we get through the thus uniquely fixed complexification operator is the relativistic neutral free quantum field with spin zero and mass μ.

Before stating the theorem in which the criterion mentioned above is proved, let us point out that, once an element J of $\mathcal{J}_{(\mathcal{M},B)}$ is given, a (real linear) operator A on $\mathcal{M}$ is also a complex linear operator in $\mathcal{M}_J$ if and only if $[A, J] = 0$. In what follows we shall deal with operators that have this property and others that do not.

Another useful remark is the following. Let $g \mapsto S_g$ be a homomorphism of a group G into the group $\mathrm{Aut}(\mathcal{M}, B)$ (no continuity condition for $g \mapsto S_g$ is required to prove the theorem), and J an element of $\mathcal{J}_{(\mathcal{M},B)}$ such that $\forall g \in G$, $[S_g, J] = 0$. Then S_g is a complex linear bounded operator in $\mathcal{M}_J$ with domain $\mathcal{M}$; we shall denote by $(S_g)_J$ its bounded extension to $\mathcal{M}_J$, which is easily shown to be a unitary automorphism of the complex Hilbert space $\mathcal{M}_J$. The mapping S_J, defined by $g \mapsto S_J(g) := (S_g)_J$, is a unitary representation of G in $\mathcal{M}_J$, as is straightforward to check.

We can now state and prove our main result.

Theorem. Let $g \mapsto S_g$ be a homomorphism of a group G into the group Aut$(\mathcal{M}, B)$. Let J be an element of $\mathcal{J}_{(\mathcal{M},B)}$ satisfying:

(i) $\forall \mathcal{G} \in \mathrm{G}, [S_g, J]=0$,

(ii) S_J is real irreducible, i.e. no closed non-trivial real linear subspace exists in $\mathcal{M}_J$ that is left invariant by S_J.

If K is an element of $\mathcal{J}_{(\mathcal{M},B)}$ such that $\forall g \in \mathrm{G}, [S_g, K]=0$, then $K=J$.

Proof. It is convenient to divide the proof into three parts.

(*a*) Here, we gain control of the real linear operator K—we do not even know whether it is bounded—deriving a relation between J and K. In this first part we shall not fully exploit assumption (ii); indeed, we shall use just the complex irreducibility of S_J, which is implied by its real irreducibility but is in general a weaker condition.

Define the operator $A := [J, K]_+ \equiv JK + KJ$ in $\mathcal{M}_J$; A is complex linear as it commutes with J, and its domain is $\mathcal{M}$. A straightforward calculation shows that $\forall m \in \mathcal{M}, \|Am\| \geq \sqrt{2}\|m\|$ ($\|\cdot\|$ means the norm of $\mathcal{M}_J$); therefore A^{-1} exists and is a bounded operator in $\mathcal{M}_J$. Consider now the complex linear operator C, defined as the closure of A^{-1} in $\mathcal{M}_J$: C is the bounded extension of A^{-1} to $\bar{R}_A$, the closure in $\mathcal{M}_J$ of the range R_A of A. First, notice that $\bar{R}_A = \mathcal{M}_J$; in fact R_A is invariant under S_J as $\forall g \in \mathrm{G}, [S_g, A]=0$, and this implies that also $\bar{R}_A$ is invariant under S_J; from the complex irreducibility of S_J we have either $\bar{R}_A$ null or $\bar{R}_A = \mathcal{M}_J$; since A^{-1} exists, we have $\bar{R}_A = \mathcal{M}_J$. Further, notice that C commutes with S_J; in fact $\forall g \in \mathrm{G}$, from $[S_g, A]=0$ we derive $A^{-1}S_g \restriction R_A = S_g A^{-1}$ (where $\restriction R_A$ means restriction to R_A); moreover, $C(S_g)_J$ is the bounded extension to $\mathcal{M}_J$ of the left-hand side of this equality and $(S_g)_J C$ is the same for the right-hand side; therefore $C(S_g)_J = (S_g)_J C$. Thus, because of the complex irreducibility of S_J and the Schur theorem, C is a complex multiple of the identity operator on $\mathcal{M}_J$, and this implies that a non-null complex number α exists such that $A = \alpha \mathbb{1}$ (the identity operator on $\mathcal{M}$).

It is now easy to check that $\forall m \in \mathcal{M}, \mathrm{Im}(Am|m)=0$ ($(\cdot|\cdot)$ means the inner product of $\mathcal{M}_J$); therefore A is a symmetric operator in $\mathcal{M}_J$, whence α is a real number. We have thus proved that a non-null real number exists such that the relation

$$KJ + JK = \alpha \mathbb{1} \tag{1}$$

between J and K holds.

(*b*) We prove here an easy lemma.

Lemma. Under the same assumptions as in the theorem, but for the possibility of weakening the real irreducibility of S_J to complex irreducibility, the following three statements are equivalent.

(i) $[K, J]=0$,

(ii) $K=J$,

(iii) $\alpha=-2$, where α is the real number that appears in the relation (1) above.

Proof of the lemma. The assumptions of the theorem (with the possibility of weakening the real irreducibility of S_J to complex irreducibility) play a role only inasmuch as from them relation (1) above follows. Indeed, the proof follows from (1) and from the properties of the complexification operators. (i)$\Rightarrow$(ii): from (i) and (1), $-2J=\alpha K$ follows, whence $K = \pm J$; as $K=-J$ is impossible, (ii) follows. (ii)$\Rightarrow$(iii): substitute K for J in (1). (iii)$\Rightarrow$(i): a lengthy but straightforward calculation, in which several times use of (1) is made, leads to the equality

$$\forall m \in \mathcal{M}, \|[J, K]m\|^2 = (\alpha^2 - 4)\|m\|^2, \tag{2}$$

which shows that (i) follows from (iii) and completes the proof of the lemma.

(*c*) Now, we conclude the proof of the theorem; here full use is made of assumption (ii).

First, observe that $\alpha^2 - 4 \geqslant 0$ follows from relation (2) above. Now, define the operator $T := (\alpha^2-4)^{1/2}J - [K, J]$ in $\mathcal{M}_J$, which is *a priori* only real linear and whose domain is $\mathcal{M}$. From (2) above we see that $[K, J]$ is a bounded real linear operator in $\mathcal{M}_J$; thus, T is bounded and we can consider its bounded extension $\bar{T}$ to $\mathcal{M}_J$. A straightforward computation, in which use is made of relation (1) derived in part (*a*) and of the equality $J^2 = -\mathbb{1}$, shows that $T^2 = 0$ (the null operator on $\mathcal{M}$), whence also $\bar{T}^2 = 0$ (the null operator on $\mathcal{M}_J$). Notice now that $\forall g \in \mathrm{G}, [S_g, T] = 0$ by the assumptions of the theorem and therefore, by an argument already used in part (*a*), $\bar{T}(S_g)_J = (S_g)_J\bar{T}$. This implies that the kernel $\mathrm{Ker}_{\bar{T}}$ of $\bar{T}$, which is a closed real linear subspace of $\mathcal{M}_J$, is left invariant by S_J, and therefore either $\mathrm{Ker}_{\bar{T}} = \mathcal{M}_J$ or $\mathrm{Ker}_{\bar{T}}$ is null; but this last condition cannot hold, because $\bar{T}^2 = 0$; therefore $\bar{T} = 0$, whence $T = 0$, i.e. $(\alpha^2-4)^{1/2}J = [K, J]$. Notice now that $[[K, J], J]_+ = 0$ follows from $J^2 = -\mathbb{1}$; thus $(\alpha^2-4)^{1/2} = 0$, whence $[K, J] = 0$. From this equality and from the lemma proved in (*b*) we get $K = J$. The proof of the theorem is complete.

As we have just seen, the real irreducibility of S_J (as opposed to the weaker complex irreducibility) is used only in (*c*) of the proof of the theorem. However, its role is crucial and if it was altogether replaced by complex irreducibility the theorem would be false, as the following easy example shows (Gallone and Sparzani 1979). Take $\mathcal{M} = \mathbb{R}^2$ and B the canonical symplectic form on $\mathbb{R}^2$; for every triple $(\gamma, \eta, \rho) \in \mathbb{R}^3$ the mapping

$$\mathbb{R} \ni t \mapsto S_t^{(\gamma,\eta,\rho)} := \begin{vmatrix} \cos\omega t - 2\rho(\sin\omega t)/\omega & 2\gamma(\sin\omega t)/\omega \\ -2\eta(\sin\omega t)/\omega & \cos\omega t + 2\rho(\sin\omega t)/\omega \end{vmatrix},$$

where

$$\omega := \begin{cases} 2(\eta\gamma - \rho^2)^{1/2} & \text{if } \eta\gamma - \rho^2 \geqslant 0 \\ 2\mathrm{i}(\rho^2 - \eta\gamma)^{1/2} & \text{if } \eta\gamma - \rho^2 < 0 \end{cases}$$

and $(\sin\omega t)/\omega$ means t when $\omega = 0$, is a continuous symplectic representation of the group $\mathbb{R}$ in this symplectic space, and there is no other continuous symplectic representation of $\mathbb{R}$. A complexification operator exists that commutes with $S_t^{(\gamma,\eta,\rho)}$ if and only if either $\gamma = \eta = \rho = 0$ or $\gamma\eta - \rho^2 > 0$. In the latter case, $t \mapsto S_t^{(\gamma,\eta,\rho)}$ is real irreducible and indeed—in agreement with the theorem—there is just one complexification operator that commutes with it. On the other hand, $t \mapsto S_t^{(0,0,0)}$ is the trivial symplectic representation and commutes with all the (infinitely many) complexification operators; however, it is complex irreducible (but not real irreducible!) in $\mathcal{M}_J$ for any complexification operator J. This shows that the condition of real irreducibility in the theorem cannot be weakened to complex irreducibility.

As a further comment, it is worth mentioning that in the course of the proof we had to be careful in dealing with the complexification operator K also because an element of $\mathcal{J}_{(\mathcal{M},B)}$ may happen not to be continuous in the topology defined over $\mathcal{M}$ by another element of $\mathcal{J}_{(\mathcal{M},B)}$, as the following example shows. Let $\mathcal{H}$ be a complex separable Hilbert space and $\{u_n\}_{n\in N}$ an orthonormal basis in $\mathcal{H}$; $\{u_n, \mathrm{i}u_n\}_{n\in N}$ is a real orthonormal basis. On the algebraic linear subspace $\mathcal{D}$ spanned by this basis, define the real linear operator J_1 setting $J_1 u_n = -\mathrm{i}n u_n$ and $J_1(\mathrm{i}u_n) = (1/n)u_n$; it is easy to see that—over the symplectic space $(\mathscr{D}, B)$, where B is the imaginary part of the inner product—J_1 is a complexification operator and so is the operator J_2 defined as the multiplication by i. Now, J_1 is not continuous in $\mathscr{D}_{J_2} = \mathscr{H}$ and $\mathscr{D}_{J_1} \neq \mathscr{D}_{J_2}$. Of course, *after proving the theorem*, we know that the complexification operator K of the theorem is well behaved in $\mathscr{M}_J$: indeed, K coincides with J!

We shall now make use of the uniqueness result of the theorem to prove that the relativistic free neutral scalar field with positive mass admits of a unique Segal quantisation. In the symplectic space $(\mathscr{M}, B)$ that describes the classical kinematics of this system, $\mathscr{M}$ is a set of real solutions to the positive mass μ and spin zero Klein–Gordon equation $(\Box + \mu^2)\phi(\boldsymbol{x}, x_0) = 0$; the symplectic form B is the Wronskian of two solutions

$$B(\phi, \psi) := \int_{x_0=0} (\dot{\phi}\psi - \phi\dot{\psi})\, \mathrm{d}\boldsymbol{x},$$

which means B is the Poisson bracket of the classical system with canonical variables $\phi(\boldsymbol{x}) := \phi(\boldsymbol{x}, 0)$, $\pi(\boldsymbol{x}) := \dot{\phi}(\boldsymbol{x}, 0)$; the form B is uniquely determined, up to a scalar factor, by the condition that it be invariant under the natural action of the restricted Poincaré group $\mathscr{P}_+^\uparrow$ on $\mathscr{M}$:

$$\phi(x) \mapsto (S_{(a,\Lambda)}\phi)(x) := \phi(\Lambda^{-1}(x-a))$$

where $(a, \Lambda) \in \mathscr{P}_+^\uparrow$. A complexification operator over $(\mathscr{M}, B)$ is defined by the mapping

$$(\dot{\phi}, \phi)_{x_0=0} \mapsto (D\phi, -D^{-1}\dot{\phi})_{x_0=0}$$

where $D := (-\Delta + \mu^2)^{1/2}$; in fact this defines an operator J on $\mathscr{M}$, since the pair $(\phi, \dot{\phi})_{x_0=0}$ determines a unique solution of the wave-equation, and also $J \in \mathscr{J}_{(\mathscr{M},B)}$ can be easily proved†. Now, $\mathscr{P}_+^\uparrow$ can be considered a classical symmetry group for the free field since $(a, \Lambda) \mapsto S_{(a,\Lambda)}$ is a continuous symplectic representation in $(\mathscr{M}, B)$ of $\mathscr{P}_+^\uparrow$. Moreover it can be shown that $\forall (a, \Lambda) \in \mathscr{P}_+^\uparrow$, $[S_{(a,\Lambda)}, J] = 0$; therefore, through the second quantisation defined by J, both the classical kinematical description based on $(\mathscr{M}, B)$ and the classical symmetry group $(a, \Lambda) \mapsto S_{(a,\Lambda)}$ can be quantised. Indeed, the quantum system that is thus obtained is the relativistic neutral free quantum field with spin zero and mass μ.

The problem naturally arises whether this quantum system is the only one that can be obtained through second quantisation from $(\mathscr{M}, B)$ and $S_{(a,\Lambda)}$, i.e. whether an element K of $\mathscr{J}_{(\mathscr{M},B)}$ that commutes with $S_{(a,\Lambda)}$ for each (a, Λ) must coincide with J. The theorem stated above proves that this is indeed the case, since the unitary representation S_J of $\mathscr{P}_+^\uparrow$ in $\mathscr{M}_J$ defined by $S_{(a,\Lambda)}$ and J is real irreducible. This can be either seen directly or, using a result of Weinless (1969, remark 1.5), derived from the fact that S_J is complex irreducible and the self-adjoint generator of the time translations in S_J is

† From a purely classical point of view, several spaces of solutions could be chosen for $\mathscr{M}$ in an equally natural way. Say, the space $\mathscr{M}'$ of solutions whose Cauchy data are infinitely differentiable and have compact support, or the space $\mathscr{M}''$ of solutions whose Cauchy data have Fourier transforms with these properties ($\mathscr{M}''$ is the space of the so-called regular wave-packets). However, our prescription for J (which actually comes from fore-knowledge of the quantum situation) does not make sense on $\mathscr{M}'$, while it does on $\mathscr{M}''$, which can therefore be taken to be $\mathscr{M}$. Of course, there are other choices for $\mathscr{M}$ (e.g. the Hilbert space $\mathscr{M}_J''$ or its subspace of the 'finite energy solutions') that make B and J well defined. This is an example of what is described in the footnote in § 1.

strictly positive: indeed, S_J is (a realisation of) the mass μ and spin zero continuous unitary irreducible representation of $\mathscr{P}_+^\uparrow$.

It is interesting to compare our result with that of Kay (1979). The well known complexification of the free field has three properties.

(i) Time translations are represented as a unitary group.

(ii) The whole Poincaré group is represented (real irreducibly) as a unitary group.

(iii) Positive energy.

Essentially, Kay shows that (i) and (iii) alone imply uniqueness (and hence (ii)). Our result is that (i) and (ii) alone imply uniqueness (and hence (iii)).

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